

**Local and Global Hartogs-Bochner Phenomenon in Tubes**

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**Abstract**

A generalization of the Hartogs theorem is proved for a class of Tube structures  $(M, G, \mathcal{V})$ . We assume that the intervening commutative Lie algebra  $G$  admits at least  $\text{codim } \mathcal{V}$  globally solvable generators. We give necessary and sufficient conditions for triviality of the first cohomological group with compact support associated to the Tube structure to be trivial. A such global result was previously obtained only when  $M = \mathbf{R}^n \times \mathbf{R}^m$  with  $\partial/\partial x_j$  for  $j = 1, \dots, m$  generating a Lie subalgebra of  $G$ .

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### **Introduction.**

We start recalling the so called Bochner's extension theorem ([Bo1,2]). It states that if  $u$  is a holomorphic function defined in an open connected set  $\mathbf{R}^m + i\Omega \subset \mathbf{C}^n$  then it extends as a holomorphic function to the linear convex envelope  $\mathbf{R}^m + i\widehat{\Omega}$  of  $\mathbf{R}^m + i\Omega$  (one year before that Stein ([St]) proved this result for  $n = 2$ ). A kind of local version of the Bochner extension theorem is found in Komatsu [Ko] where  $\mathbf{R}^m$  is replaced by a ball  $B_R$  centered at the origin with radius  $R$ . Later Andronikof ([An]) precise the dependence between  $R$  and the size domain of the extension, namely:

*Let  $\Omega \subset \mathbf{R}^n$  be a convex bounded set of dimension  $> 2$ ; if*

$$R - \rho > \sqrt{2} \operatorname{diameter}(\Omega)$$

*then each function holomorphic on a neighborhood of the tube  $B_R \times \partial\Omega$  has a unique holomorphic extension to a neighborhood of the tube  $B_\rho \times \Omega$ . An example of Ye ([Ye]) shows that  $R$  is necessarily bigger than  $(1/2)\operatorname{diameter}(\Omega)$ , leaving the question of finding the sharp constant in the interval  $(1/2, \sqrt{2}]$ . Another classical extension theorem is due to Hartogs([Har1,2]) and it asserts that a holomorphic function in  $\mathbf{C}^n \setminus \Omega$ , where  $\Omega$  is a bounded open domain with connected boundary  $\partial\Omega$  extends itself to all of  $\mathbf{C}^n$  as a holomorphic function. The Bochner extension theorem implies the Hartogs's one as we see now; let*

$\mathbf{C}^n \xrightarrow{\Pi} \mathbf{R}^n$  be the projection into the imaginary part;  $\Pi(x + it) = t$ . Then

$$\mathbf{C}^n \setminus \overline{\Omega} \supset \mathbf{R}^n + i \mathbf{R}^n \setminus \Pi(\overline{\Omega}) \quad (0.1)$$

Since the convex envelope of  $\mathbf{R}^n \setminus \Pi(\overline{\Omega})$  is  $\mathbf{R}^n$  the Hartogs extension theorem follows for pairs  $(\Omega, K)$  with  $\Omega \setminus K$  connected. Only four years after Fichera ([Fi]) published his work reducing the amount of CR data to  $\partial\Omega$  under certain regularity constraints, Ehrenpreis ([Eh]) gave a new proof of the Hartogs extension theorem. The proof of Ehrenpreis was remarkable simple and its main idea is a cohomological vanishing argument. The same idea applied by Hounie & Tavares ([HT]) to gives necessary and sufficient conditions for the validity of the Fichera's version of the Hartogs extension theorem for a smooth globally integrable Tubes structures in  $\mathbf{R}^m \times \mathbf{R}^n$ . By a smooth globally integrable Hypoanalytic Tubes structures in  $\mathbf{R}^m \times \mathbf{R}^n$ , we mean a subbundle  $\mathcal{L} \subset \mathbf{C} \otimes T(\mathbf{R}^m \times \mathbf{R}^n)$  such that  $\mathcal{L}_p = \text{Ker } dZ(p)$  where  $Z : \mathbf{R}^m \times \mathbf{R}^n \longrightarrow \mathbf{C}^m$  is a smooth function  $Z(x, t) = x + i\Phi(t)$ . It extends the concept of the Cauchy-Riemann system in  $\mathbf{C}^m$ . By a hypoanalytic structure we understand as a pair  $(M, \mathcal{L})$  consisting of a smooth manifold  $M$  and a subbundle  $\mathcal{L} \subset \mathbf{C} \otimes TM$  endowed with a associated hypoanalytic atlas  $(U_\alpha, Z_\alpha)$ . We mean  $\cup_\alpha U_\alpha = M$  and the maps

$$Z_\alpha : U_\alpha \longrightarrow \mathbf{C}^m \text{ with } m = \dim M - \dim_{\mathbf{C}} \mathcal{L} \quad (0.2)$$

are smooth and  $\det dZ_\alpha \neq 0$  and if  $p \in U_\alpha$  then  $\mathcal{L}_p = \ker Z_\alpha(p)$ . Finally the etymology comes from the constrain that  $Z_\beta = Z_\alpha \circ H_{\alpha, \beta}$  in an neighborhood every point  $p \in U_\alpha \cap U_\beta$  where  $H_{\alpha, \beta}$  is a biholomorphism in some open neighborhood of  $Z_\beta(p)$ . It is well known that *fibers* of the hypoanalytic structure defined by the germs

$$\mathcal{F}(p) = \mathcal{C}_{Z_\alpha}(p) = \{Z_\alpha = Z_\alpha(p)\} \quad (0.3)$$

are hypoanalytic invariants of the structure. The Sussmann's orbit  $\mathcal{O}_{\mathcal{L}}(p)$  (named after Sussmann ([Su])) is the minimal smooth submanifold contain-

ing  $p$  which supports  $\mathcal{L}$  in its complexified tangent space. We say that a smooth germ of function  $u$  at  $p$  is *hypoanalytic* if  $du$  is a germ of a section of  $\mathcal{L}^\perp$ . If  $\mathcal{O}_{\mathcal{L}}(p)$  is compact then the trace of a hypoanalytic function in the orbit must be constant otherwise

An Tube structure  $(M, \mathcal{L}, \mathcal{G})$  is a hypoanalytic structure endowed with a commutative Lie algebra  $\mathcal{G} \subset TM$  which verifies the conditions:

- <sub>1</sub> if  $\mathcal{A}_p \subset T_p M$  is the span of  $\mathcal{G}_p$  then  $\dim \mathcal{A}_p \geq \text{codim } \mathcal{L}$  for all  $p \in M$ ,
- <sub>2</sub>  $\mathcal{L}_p + \mathbf{C} \otimes \mathcal{G}_p = \mathbf{C} \otimes T_p M$  for all  $p \in M$ ,
- <sub>3</sub>  $[\mathcal{L}, \mathcal{G}] \subset \mathcal{L}$ .

It follows from •<sub>1</sub> that  $m = \dim \mathcal{G}$  is well defined and greater or equal to

$$\text{codim}_{\mathbf{C}} \mathcal{L} = \dim M - \dim_{\mathbf{C}} \mathcal{L}.$$

Under these hypothesis one can always find an hypoanalytic atlas  $(U_\alpha, Z_\alpha)$  such that  $Z_\alpha(x, t) = x + \Phi(t)$  for suitable coordinates where  $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}$  is a subset of generators of  $A$  over  $U_\alpha$ .

Let us denote by  $\mathcal{F}_Z(p)$  the germ of the closed set  $\{Z = Z(p)\}$  for an arbitrary hypoanalytic function  $Z$  at  $p \in M$ . For arbitrary Tubes structures  $(M, \mathcal{L}, \mathcal{G})$  we have the following characterization of the *local* Hartogs property;

**Theorem.A.** A Tube  $(M, \mathcal{L}, \mathcal{G})$  has the *local* Hartogs property if and only if  $\mathcal{F}_Z(p)$  is connected for all hypoanalytic germs  $Z$  at  $p$  for all  $p \in M$ .

**Remark.** Recently was established in work of Henkin & Michel ([HM]) for abstract real analytic (CR)-structures  $(M, \mathcal{L})$  that the *local* Hartogs phenomenon is equivalent to  $(M, \mathcal{L})$  be nowhere strictly pseudoconvex with  $\dim M \geq 3$ . Actually the concept of pseudoconvexity is belongs to the larger class of structures called hypoanalytic structures. The Levi form  $\Xi_\theta(p)$  is a hypoanalytic invariant defined in  $\mathcal{L}_p \times \mathcal{L}_p$  for every  $\theta \in \Sigma_p = \mathcal{L}^\perp \cap T_p^* M$  by

$$\Xi_p^\theta(v, w) = \theta([Re L_0, Im L_1])(p).$$

Here  $L_0, L_1$  are germs sections of  $\mathcal{V}$  satisfying  $L_0(p) = v$  and  $L_1(p) = w$  and  $\Sigma_p$  is the characteristic set of  $(M, \mathcal{L})$ . Actually it is an well defined object of the Sussmman orbit  $\mathcal{O}_{\mathcal{L}}(p)$  of  $\mathcal{V}$ . We say that  $(M, \mathcal{L})$  is strictly pseudoconvex at  $p \in M$  if  $\Xi_p^\theta$  is non degenerated with all the eigenvalues with a same sign. Consequently  $\mathcal{L}_p \subset \mathbf{C} \otimes T_p \mathcal{O}_{\mathcal{L}}(p)$  and when  $\Xi_\theta(p)$  is nondegenerated with all eigenvalues of a same sign we say that  $\mathcal{L}$  is strictly pseudoconvex at  $\theta \in \Sigma_p$ . When this happens we can always find a germ of a hypoanalytic function at  $p$  such that  $dZ(p) \neq 0$  such that  $\{Z = Z(p)\} = \{p\}$ . Thus being nowhere strictly pseudoconvex is necessary condition for a hypoanalytic structure verify the local Hartogs property.

We now adress the question of whether the *global* Hartogs property holds for all pairs  $(K, U)$  of compact sets  $K \subset U$  where  $U \subset M$  is open. Hopefully we answer the question of Nacinovitch and Hill about the example of the CR-structure on the hypersurface  $|z_1|^2 + |z_2|^2 - |z_3|^2 = 1$  in  $\mathbf{C}^3$  where the zero of the restriction of  $z_3$  becomes compact failing the *global* Hartogs property but curiously holding the *local* one. Such hypersurface is actually a zero of a homogenous solution of a Tube structure globally defined by the map

$$Z : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \longrightarrow \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times \mathbf{C}$$

where  $Z(v_1, v_2, v_3, v_4) = (z_1, z_2, z_3, x + i\Phi(|v_1|, |v_2|, |v_3|))$  with  $dz_1 \wedge dz_2 \wedge dz_3 \wedge dx + i d\Phi \neq 0$  on  $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ . Finally it is taken  $\Phi(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 - \xi_3^2$  and the zeroes of  $\{x + i\Phi(|v_1|, |v_2|, |v_3|)\}$  becomes CR-substructures which are actually also Tubes. By means of a right biholomorphism we find  $Z(v_1, v_2, v_3, v_4) = (z_1, z_2, z_3, x + i\Phi(\text{Im } z_1, \text{Im } z_2, \text{Im } z_3))$  thus a tube according the Definition VI.9.2 in Treves([Tr1]) and the embedded CR-submanifolds  $\{x + i\Phi(|v_1|, |v_2|, |v_3|) = \text{constant}\}$  are Tubes with the restriction of  $Z_0 = (z_1, z_2, z_3)$  as a global integral. After a unitary linear right composition the intersection

zeroes  $\{x + i\Phi(\text{Im } z_1, \text{Im } z_2, \text{Im } z_3) = z_3 = a + ib\}$  have the expression

$$\{\mathbf{R}^2 \times \{(t_1, t_2) \in \mathbf{R}^2 : \text{Im}^2 z_1 + \text{Im}^2 z_2 = c + b^2\} \times \{a + ib\} \times \{a + ic\}$$

and it is empty if  $c < -b^2$ . When  $c = b^2$  it is the plane

$$\{\mathbf{R}^2 \times \{(0, 0)\} \times \{a + ib\} \times \{a + ic\}$$

becoming homeomorphic to  $\mathbf{R}^2 \times \mathbf{S}$  for all  $c > -b^2$ . It happens that the function  $\text{Re}[iz_3 + \kappa(z_1^2 + z_2^2)] = (1 - \kappa)[\text{Im}^2 z_1 + \text{Im}^2 z_2] + \kappa[\text{Re}^2 z_1 + \text{Re}^2 z_2]$  has for  $0 < \kappa < 1$  compact zeros homeomorphic to  $\mathbf{S}^3 \subset \mathbf{R}^2 \times \mathbf{R}^2 \times \{a + ib\} \times \{a + i\Phi(\text{Im } z_1, \text{Im } z_2, b) = a + ic\}$  and noncompact zeros for  $\kappa \geq 1$ .

Let us now denote by  $C_Z(p)$  the closed set  $\{\Re Z \leq \Re Z(p)\}$  for an arbitrary hypoanalytic function  $Z$ . We will say that a Tube structure  $(M, \mathcal{L}, \mathcal{G})$  verifies the *global* Hartogs condition (H) if:

- <sub>4</sub> the Lie algebra  $\mathcal{G}$  admits at least  $\text{codim } \mathcal{L}$  globally solvable generators,
- <sub>5</sub>  $C_Z(p) \subset M$  does not have compact components for all hypoanalytic function  $Z$  and  $p \in M$ ,
- <sub>6</sub>  $\mathcal{O}_{\mathcal{L}}(p) \subset M$  is never compact for all  $p \in M$ .

The example of Hill & Nacinovitch ([HN]) will show that condition •<sub>4</sub> is necessary. The condition •<sub>5</sub> is obviously needed for global Hartogs property holds. Otherwise any open set containing one compact component would fail the Hartogs property. Finally we may consider the quotient space  $\mathcal{O}_{\mathcal{L}}$  defined by the equivalent relation  $\sim$ , where  $p \sim q$  in  $M$  if and only if  $p, q \in \mathcal{O}_{\mathcal{L}}(p)$ . Then every  $u \in C(\mathcal{O}_{\mathcal{L}})$  can be lifted to a function in  $C(M)$  which is a weak solution of  $\mathcal{L}$  and  $(u - u(p))^{-1}$  fails the global hartogs property showing that •<sub>6</sub> is also necessary.

We can now state that

**Theorem.B.** Let  $M$  be simply connected and  $(M, \mathcal{L}, \mathcal{G})$  be a Tube structure. Then  $(M, \mathcal{L}, \mathcal{G})$  verify (H) if and only if *global* Hartogs property holds.

**Remark.** This gives a explanation for the embedded example of Hill & Nacinovitch (see [HN]) which gives an example of a Tube structure which verify the local Hartogs phenomena but not the global one. The Tube structure in question is defined by given by the map

$$Z : \mathbf{C}^3 \times \mathbf{R} \longrightarrow \mathbf{C}^4$$

defined by  $Z(z, y) = (z, y + \mathbf{i}(|z_1^2| + |z_2^2| - |z_3^2|))$ . By means of an biholomorphism in  $\mathbf{C}^4$  we may rewrite  $Z : \mathbf{C}^3 \times \mathbf{R} \longrightarrow \mathbf{C}^4$  as

$$Z(z, x) = (x_1 + \mathbf{i} t_1, x_2 + \mathbf{i} t_2, x_3 + \mathbf{i} t_3, x + \mathbf{i}(t_1^2 + t_2^2 - t_3^2)).$$

In this case there exists only one orbit and every the Hartogs global phenomena holds. On the other hand the zero  $\mathcal{C}_Z(p)$  with  $Z(p) = \mathbf{i}$  is a hypoanalytic submanifold which happens to be a globally integrable. The global integral in question is the restriction of  $(x_1 + \mathbf{i} t_1, x_2 + \mathbf{i} t_2, x_3 + \mathbf{i} t_3)$  to the zero

$$\mathcal{C}_Z(p) = \{x + \mathbf{i}(t_1^2 + t_2^2 - t_3^2) = \mathbf{i}\}.$$

Thus it is a Tube which enjoy the the local Hartogs phenomena but not the global one. It happens that  $(x_3 + \mathbf{i} t_3)^{-1}$  is well defined in  $\mathcal{C}_Z(p)$  with  $Z(p) = \mathbf{i}$  except by its with intersection with  $x_3 + \mathbf{i} t_3 = 0$ . The latter intersection is a set homeomorphic to the cylinder  $\mathbf{R}^2 \times \mathbf{S}^1$  and one can check that the function

$$Z_\kappa = x_3 + \mathbf{i} t_3 + \kappa((x_1 + \mathbf{i} t_1)^2 + (x_2 + \mathbf{i} t_2)^2)$$

for some small  $k < 1$  has a compact zero inside a Torus contained in  $\mathbf{R}^2 \times \mathbf{S}^1 \subset \mathbf{R}^2 \times \mathbf{i} \mathbf{R}^2 \simeq \mathbf{C}^2$  violating the condition  $\bullet_5$  in Theorem B. The characterization given in [HT] for the global Hartogs phenomena in here stands for  $\bullet_5$  one of the global condition in (H). Thus the Theorem B is a generalization of the result presented there.

## 2.Proofs of Theorem A and B

**Proof of Theorem A.** It follows from the main result in [HT] that a Tube structure  $(M, \mathcal{L}, \mathcal{G}, )$  enjoys the *local* Hartogs property if and only if the germ  $\mathcal{C}_Z(p)$  of a hypoanalytic function  $Z$  with  $dZ(p) \neq 0$  at  $p$  is connected and equivalent to nowhere strictly pseudoconvexity for Tubes structures  $(M, A, \mathcal{L})$ . Observe that  $\mathbf{C} \otimes T_p M = \mathcal{L}_p + \mathbf{C} \otimes \mathcal{A}_p \subset \mathbf{C} \otimes T_p \mathcal{O}_{\mathcal{L}} + \mathbf{C} \otimes \mathcal{A}_p$  and consequently  $\mathbf{C} \otimes N_p^* \mathcal{O}_{\mathcal{L}}(p) \subset \mathcal{L}^\perp$ . If the local Hartogs phenomena occurs for this hypoanalytic structure then the germ  $\mathcal{C}_Z(p)$  for a hypoanalytic function with  $dZ(p) \neq 0$  must be connected. Otherwise it will display some compact component or a denumerable set of components. In the first case  $Z^{-1}$  would fails the Hartogs phenomena for some pair  $(U, \mathcal{C}_Z(p) \cap U)$  and in the second is void, otherwise  $dZ(p) = 0$ . By means of a complex linear transformation one may assume that for a hypoanalytic chart  $(Z_\alpha, U)$  with  $p \in U$  that  $Z_\alpha(p) = 0$  with  $dZ_\alpha(p) = I$ . Then  $N_p^* \mathcal{O}_{\mathcal{L}}(p)$  will necessarily have a basis among the differentials  $\{d \operatorname{Re} Z_{1,\alpha}(p), \dots, d \operatorname{Re} Z_{m,\alpha}(p)\}$ . This implies with  $Z_\alpha^2 = Z_{1,\alpha}^2 + \dots + Z_{m,\alpha}^2$  and large  $\kappa$  that the germ  $\mathcal{C}_{Z+\kappa Z_\alpha^2}(p) \cap \mathcal{O}_{\mathcal{L}}(p)$  must be connected if  $\mathcal{C}_Z(p)$  is. Consequently for Tubes a necessary and sufficient condition the validity of local Hartogs phenomena is translated on germs  $\mathcal{F}_Z(p) = \mathcal{C}_Z(p) \cap \mathcal{O}_{\mathcal{L}}(p)$  by the condition;  $\mathcal{F}_Z(p)$  is connected for all hypoanalytic germs  $Z$  with  $\operatorname{Ker} dZ(p) = \{0\}$  (P).

**Proof of Theorem B.(1st version via Ehrenpreis argument )**

We will prove that the first cohomological group of the complex induced by  $\mathcal{L}$  is trivial reviving the original idea of Ehrenpreis [Eh] and giving a stronger version of Theorem.B. We select  $m = \operatorname{codim} \mathcal{L}$  globally integrable vector fields from  $\mathcal{G}$  and assume without loss of generality that  $\dim \mathcal{G} = m$ . It follows that there exist smooth manifold  $N$  such that  $M = \mathbf{R}^m \times N$ . For  $m = 1$  it is the statment of Theorem 6.4.2 (f) in [DH]. When  $m$  is bigger than one we proceed by induction taking advantage of the commutative property of the fields. As a consequence we get an open projection  $\Pi_A : M \rightarrow N$  having as fibers the the  $m$ -dimensional submanifolds  $A \subset M$  verifying  $T_p A = \mathcal{A}_p$  if  $p \in A$ , that is



$A = \mathbf{R}^m \times \{\Pi_A(p)\}$ . Now follows from the characterization of tubes structures found in VI.8 Partial Local Group Structures([Tr1]) that one can construct a hypoanalytic atlas  $(U_\alpha, Z_\alpha)$  such that  $Z_\alpha(x, t) = x + i\Phi(t)$  where the first coordinates  $x$  are first integrals of the chosen  $m$  globally integrable vector fields in  $\mathcal{G}$ . Since a pair of hypoanalytic charts  $(U_\alpha, Z_\alpha), (U_{\alpha'}, Z_{\alpha'})$  changes by a biholomorphism and they have identical real parts in  $U_\alpha \cap U_{\alpha'}$  they must agree there. It follows that  $\mathcal{L}$  has a global integral  $Z$  and the topological space  $M/\sim_{\mathcal{F}}$ , where  $\sim_{\mathcal{F}}$  is the equivalence of being in a same fiber of  $\mathcal{L}$ , is globally defined. Let  $Z = (Z^1, \dots, Z^m)$  be a global integral for  $(M, \mathcal{L}, A)$ , that is a map from  $M \longrightarrow \mathbf{C}^k$  with  $k = \text{codim } \mathcal{L}$ . Let  $\omega = \sum_{j=1}^n f_j dt_j$  be a smooth closed class in the first cohomological group with compact support induced by the differential complex associated to  $\mathcal{L}$ . We mean that

$$d\omega \wedge dZ = 0$$

where  $dZ = dZ_1 \wedge \dots \wedge dZ_m$ . The same steps in [HT] by performing Fourier transform of  $\omega \wedge dZ$  in the linear fibers  $\{t\} \times \mathbf{R}^m$  to find  $\hat{\omega} \in \wedge^1 T^*(N)$  such that

$$d_t e^{\Phi \cdot \xi} \hat{\omega} = 0 \text{ for all } \xi \in \mathbf{R}^{m*}.$$

Since  $M$  is simply connected so it is  $N$  which enables us to define  $v(\xi, t)$  by

$$d_t v(\xi, t) = e^{\Phi \cdot \xi} \hat{\omega}, \quad v(\xi, t_0) = 0$$

where  $t_0 \in N \setminus \Pi_A(\text{supp } \omega)$ . Now set  $\hat{u}(\xi, t) = e^{-\Phi \cdot \xi} v(\xi, t)$  which vanishes outside  $\Pi_A(\text{supp } \omega)$ . It remains to prove that  $\hat{u}$  is indeed the fiber Fourier transform of a function  $u \in C_c^\infty(M)$  to finishes the proof. It follows from  $\bullet_5$  that the sublevels

$$\mathcal{C}_{-iZ}(0, t) = \{s \in N : \Phi(s) \cdot \xi \leq \Phi(t) \cdot \xi\}$$

does not have compact components. Since it is a closed set it implies that  $N$  can not be compact. We now cover  $N$  by charts  $(\chi_\beta, W_\beta)$  associated to the maximal

atlas of  $N$  such that each one maps  $W_\beta$  onto  $Q_0 = [0, 1]^n$ . Also we may assume that  $\{W_\beta\}$  is a locally finite covering. Now we consider a subdivision of  $Q_0$  in  $2^{nk}$  cubes  $Q_k$  of side length  $2^{-k}$ . Then any polygonal line inside  $Q_0$  which intercepts each division cube in a unique line segment will have a length bounded by  $\sqrt{n}2^{-k}2^{nk} = \sqrt{n}2^{(n-1)k}$ . In particular the image of a such polygonal line by  $\chi_\beta^{-1}$  into  $W_\beta$  will have length bounded by  $C_\beta\sqrt{n}2^{(n-1)k}$  for some metric in  $N$  which is equivalent to the euclidian metric of  $Q_0$  via any  $\chi_\beta$ . We now consider only cubes  $Q_k$  such that  $\chi_\beta(Q_k)$  meets the *connected* component of  $\mathcal{C}_{-iZ}(0, t)$  which contains  $t$  for some  $\beta$ . It entails that  $\cup_\beta \chi_\beta(Q_k) \supset \mathcal{C}_{-iZ}(0, t)$  is a connected set and we can find a curve differentiable by parts  $\gamma$  linking  $t$  to an arbitrary point in  $\cup_\beta \chi_\beta(Q_k)$  such that  $\chi_\beta(\gamma)$  is a polygonal curve in  $[0, 1]^n$  which meets any  $Q_k$  in a line segment for all  $\beta$ . Now every  $s \in \gamma$  is at a distance (for the chosen metric) comparable with  $\sqrt{n}2^{-k}$  from the component of  $\mathcal{C}_{-iZ}(0, t)$  which contains  $t$ . Let  $t'$  a point of the component within this range and apply the mean value theorem to obtain

$$|\Phi(s) - \Phi(t') \cdot \xi| \leq \sup_{t' \in Q_k} |\nabla \Phi(t')| \sqrt{n}2^{-n} |\xi|.$$

It is also true that  $(\Phi(s) - \Phi(s')) \cdot \xi \leq 0$  for every  $\xi \in \{t\} \times \mathbf{R}^{m*}$  with  $t \in N$  and we can choose  $\gamma$  such that  $\partial Q_k \cap \chi_\beta(W_\beta \cap \gamma)$  oriented set  $\{t_0, t_1\}$  obeying  $|t_0 - t|$  and  $|t_1 - t|$  are minimum and maximum of  $|s - t|$  with  $s \in \gamma \cap Q_k$ . Now we may estimate  $\hat{u}$  as  $|\hat{u}(\xi, t)| =$

$$\left| \int_\gamma e^{(\Phi(s) - \Phi(t)) \cdot \xi} \hat{\omega} \right| \leq \left| \int_\gamma e^{(\Phi(t') - \Phi(t)) \cdot \xi} \hat{\omega} \right| \leq \sqrt{n}2^{(n-1)k} e^{C|\xi|2^{-k}} \sup |\hat{\omega}|$$

where the supreme of  $|\hat{\omega}|$  is uniformly bounded in  $\Pi_A(\text{supp } \omega)$  by multiples of arbitrary powers of  $(1 + |\xi|)^{-1}$ . Choosing  $2^{-k}$  comparable with  $(1 + |\xi|)^{-1}$  we may find constants such that

$$|\hat{u}(\xi, t)| \leq C_l (1 + |\xi|)^{-l} \text{ for } t \in \Pi_A(\text{supp } \omega), \xi \in \mathbf{R}^m, l \in \mathbf{N}.$$

This happens because  $\hat{u}(\xi, t)$  is uniformly bounded in the Schwartz space  $\mathcal{S}(\mathbf{R}^m)$  for every  $t \in \mathbf{N}$ . It entails that  $u(x, t)$ , the Fourier inverse transform of  $\hat{u}(\xi, t)$  is indeed a function in  $C^\infty(M)$ . Compactness of  $\text{supp} u$  follows from a theorem of propagations of zeroes of solutions for the sections of  $\mathcal{L}$ . It states that solutions which vanishes in a neighborhood of a point  $p \in \mathcal{O}_{\mathcal{L}}(p)$  must vanishes in all orbit (see Theorem 1.1 in [HP]). In our case we consider the structure  $(M \setminus \text{supp} u, \mathcal{L}, A)$  to apply the cited theorem. Uniqueness of the solution  $u$  follows in a similar argument.

**Proof of Theorem B.(2nd version via Arens-Royden theorem)** We select  $m = \text{codim} \mathcal{L}$  globally integrable vector fields from  $\mathcal{G}$  and assume without loss of generality that  $\dim \mathcal{G} = m$ . It follows that there exist smooth manifold  $N$  such that  $M = \mathbf{R}^m \times N$ . For  $m = 1$  it is the statment of Theorem 6.4.2 (f) in [DH]. When  $m$  is bigger than one we proceed by induction taking advantage of the commutative property of the fields. As a consequence we get an open projection  $\Pi_A : M \rightarrow N$  having as fibers the the  $m$ -dimensional submanifolds  $A \subset M$  verifying  $T_p A = \mathcal{A}_p$  if  $p \in A$ , that is  $A = \mathbf{R}^m \times \{\Pi_A(p)\}$ . Now follows from the characterization of tubes structures found in VI.8 Partial Local Group Structures([Tr1]) that one can construct a hypoanalytic atlas  $(U_\alpha, Z_\alpha)$  such that  $Z_\alpha(x, t) = x + i\Phi(t)$  where the first coordinates  $x$  are first integrals of the chosen  $m$  globally integrable vector fields in  $\mathcal{G}$ . Since a pair of hypoanalytic charts  $(U_\alpha, Z_\alpha), (U_{\alpha'}, Z_{\alpha'})$  changes by a biholomorphism and they have identical real parts in  $U_\alpha \cap U_{\alpha'}$  they must agree there. It follows that  $\mathcal{L}$  has a global integral  $Z$  and the topological space  $M / \sim_{\mathcal{F}}$ , where  $\sim_{\mathcal{F}}$  is the equivalence of being in a same fiber of  $\mathcal{L}$ , is globally defined.

Let  $Z$  be a global integral for  $(M, \mathcal{L}, A)$ , that is a map from  $M \rightarrow \mathbf{C}^k$  with  $k = \text{codim} \mathcal{L}$ . Now, under the hypothesis (P) the closed set  $\mathcal{Z}(p) = \{Z = Z(p)\}$  is locally connected and the map  $\Pi_{\mathcal{L}} : M / \sim_{\mathcal{L}} \hookrightarrow Z(M)$  is relatively open and locally injective. Here  $\sim_{\mathcal{L}}$  represents the equivalence relation of two points of

$M$  being in a same component of the closed set  $\mathcal{Z}(p)$ , that is the set  $\mathcal{Z}(p)$  agree locally with  $\mathcal{F}_p$  implying that  $\mathcal{Z}(p) = \cup_{p \in \mathcal{Z}(p)} \mathcal{F}_p$ , thus invariantly defined. We call  $M / \sim_{\mathcal{L}}$  the reduced manifold by  $\mathcal{L}$  which makes any automorphism commute with a homeomorphism of  $M / \sim_{\mathcal{L}}$  via the canonical projection. Such subgroup of homeomorphisms is a hypoanalytic invariant since the germ of the fiber  $\mathcal{F}(p)$  propagates through  $\mathcal{Z}(p)$ . Thus we may say that the fiber of  $\mathcal{L}$  is globally defined and  $M / \sim_{\mathcal{L}}$  is invariant under automorphism of the structure. We mean by global diffeomorphisms of  $M$  which leaves  $\mathcal{L}$  invariant in the sense that its differential is an automorphism of  $\mathcal{L}_p$  for every  $p \in M$ . We say that an open subset  $U \subset M$  is a domain for  $\mathcal{L}$  if the canonical projection  $\Pi_{\mathcal{L}} : M \rightarrow M / \sim_{\mathcal{L}}$  is injective in  $U$ . If the intersection  $\mathcal{C}_Z(p) \cap \mathcal{O}_{\mathcal{L}}(p)$  is relatively open in  $\mathcal{O}_{\mathcal{L}}(p)$  then by uniqueness (see [Tr1])  $\mathcal{O}_{\mathcal{L}}(p) \subset \mathcal{C}_Z(p)$  and the germ propagates into the orbit  $\mathcal{O}_{\mathcal{L}}(p)$ . Despite the discreteness of fibers of the canonical projection  $\Pi_{\sim_{\mathcal{L}}}$ , one can not expect that  $M / \sim_{\mathcal{L}}$  evenly covers  $Z(M)$  and in this way not necessarily a covering space. Now, for any compact subset  $K \subset M / \sim_{\mathcal{L}}$  we consider its  $\mathcal{L}$ -convex envelope  $\widehat{K} \subset M / \sim_{\mathcal{L}}$  with respect to the finitely generated Banach algebra  $\mathcal{A}_{\mathcal{L}}(K)$  of continuous functions  $u$  of  $K$  which are uniform limits in  $K$  of polynomials in  $Z$ . Such continuous are of course also defined in  $\widehat{K}$  the polynomial convex envelope of  $K$  and  $Z(\widehat{K}) = \widehat{Z(K)}$ . Thus  $Z(\widehat{K})$  indeed agree with the maximal ideal space of the algebra  $\mathcal{A}_{\mathcal{L}}(K)$  (see Theorem 3.1.15 in ([Ho])). If the first Čech cohomology group of  $H^1(K / \sim_{\mathcal{L}})$  is not trivial it follows from the Arens-Royden theorem (see [Ar], [Ro]) that we can find a hypoanalytic polynomial  $Z_0$  such that  $dZ_0/Z_0 \neq 0$  is well defined in  $M$  and a representant of a non trivial class in  $H_{\mathbb{A}}^1(M)$  (the first cohomological DeRham group of the complex defined by the exterior derivative  $d$ ). If the DeRham cohomology group  $H_{\mathbb{A}}^1(U \setminus K)$  is trivial then  $K \subset M$  is an irremovable singularity of the ring  $\mathcal{A}(M)$  because in this case  $\text{Log } Z_0$  will be a hypoanalytic function which is defined in  $M \setminus K$  which cannot be extended for all  $M$  failing the Hartogs phenomena for

the pair  $(M, K)$ . On the other hand it follows from Poincaré duality that

$$\lim_{K \subset \subset M} H_d^p(M, M \setminus K) \simeq H_{dc}^p(M) \simeq H_d^{\dim M - p}(M).$$

Since in paracompact differentiable manifolds Čech, singular and De Rham cohomology agree and  $M/\sim_{\mathcal{L}}$  inherits from the manifold  $M$  a CW-complex structure. It follows that the Čech and singular cohomology of  $M/\sim_{\mathcal{L}}$  are well defined and agree. If  $\sim_{\mathcal{L}}$  is *proper* then there exist a natural injection

$$H_c^p(\sim_{\mathcal{L}}) : H_c^p(M/\sim_{\mathcal{L}}) \hookrightarrow H_c^p(M) \simeq \check{H}_c^p(M)$$

given by the singular cohomology functor. With  $m + n = \dim M$  we have

$$H_c^{m+1}(M/\sim_{\mathcal{L}}) \simeq H_{n-1}(M/\sim_{\mathcal{L}})$$

where  $n = \dim \mathcal{L}$  by Poincaré duality. Now we have direct decomposition

$$H_{dc}^{m+1}(M) \simeq H_c^{m+1}(\sim_{\mathcal{L}})[H_c^{m+1}(M/\sim_{\mathcal{L}})] \oplus \text{Ker } \wedge \Omega$$

where  $\Omega = d\zeta$  is a exact nonvanishing section of  $\wedge^m \mathcal{L}^\perp$  and the

$$\wedge \Omega : \wedge^1 T^*(M) \longrightarrow \wedge^{m+1} T^*(M)$$

is defined for  $\omega \in \wedge_c^1 T^*(M)$  by  $\omega \wedge \Omega$  verifies  $\Omega \wedge d = d \wedge \Omega$  and induces homomorphism  $\wedge \Omega : H_{dc}^1(M) \rightarrow H_{dc}^{m+1}(M)$ . We can represent  $H_{dc}^1(M)$  (where  $d_{\mathcal{L}}$  is the exterior derivative induced by  $d$  in the sections of  $\mathbf{C} \otimes T^*M/\mathcal{L}^\perp$ ) as the kernel of the map  $\omega \mapsto \omega \wedge \Omega$  in  $H_{dc}^1(M)$ . Thus  $\omega \wedge \Omega$  represent a class in  $H_{dc}^{m+1}(M)$  if  $\omega$  is represents a class in  $H_{dc}^1(M)$ . In this setting the Hartogs phenomena holds if and only if for all  $\omega \in H_{dc}^1(M)$  there exist  $u \in C_c^\infty(M)$  such that

$$du \wedge \Omega = \omega \wedge \Omega \tag{*}$$

Solvability of  $(*)$  assures the triviality of the intersection  $H_{dc}^{m+1}(M) \cap H_{d_{\mathcal{L}}}^1(M) = H_{d_{\mathcal{L}}}^1(M)$  which in turn must represent some subgroup of the de Rham group

$H_d^{n-1}(M)$  via Poincaré duality. The existence of a Lie algebra  $A$  oriented by  $\Omega$  allows one to decompose  $\mathcal{L} \subset \mathbf{C} \otimes A \oplus TB$  where  $B = \Pi_A(M)$  is a real  $n$ -dimensional manifold obtained by identifying the fibers of  $\Pi_A$  to points in  $M$ . It follows that every *real* section of  $TB$  has a unique lifting to  $\mathcal{L}$ . This enables us to define the connection

$$\nabla_T L(p) = T(L)(T(p)) - T_h(L(p)) \in \mathbf{C} \otimes A_p$$

where  $T\Pi_A(T_h) = T$  at  $p$  when the fibers  $\mathcal{A}_p$  of  $\Pi_A$  have a affine linear structure, and this is always the case for a open covering  $U_\alpha$  of  $M$  such that  $A$  admits  $m-1$  globally solvable generators in  $\Pi_A^{-1}(\Pi_A(U_\alpha))$ , turning  $M$  into a real vector bundle by defining local charts  $\Pi_A^{-1}(\Pi_A(U_\alpha)) \simeq \mathbf{R}^m \times \Pi_A(U_\alpha)$ . Assume that  $H_d^{m+1}(M) \simeq H^{m+1}(M)$  verifies  $H_d^{m+1}(M) = \{0\}$  which means that any section  $\omega \wedge \Omega$  is automatically exact if it represents a class in  $H_{d_c}^1(M)$ . Then we can find a section  $e\Omega + \lambda$  of  $\wedge^m T^*(M)$  such that  $d(e\Omega + \lambda) = \omega \wedge \Omega$ . It follows from the Stoke's Theorem that for for rectifiable  $m+1$ - rectifiable chain of form  $\sigma = \Pi_A^{-1}(\Pi_A(\sigma)) = \mathbf{R}^m \times \Pi_A(\sigma)$  that

$$\int_{\partial\sigma} e\Omega = \int_{\partial\sigma} (e\Omega + \lambda) = \int_{\sigma} \omega \wedge d\Omega = \int_{t \in \Pi_A(\sigma)} \int_{\Pi^{-1}(t)} \omega \wedge \Omega = \int_{\Pi_A(\sigma)} \int_{\mathbf{R}^m} \omega \wedge \Omega$$

for all  $\omega \in H_{d_c}^{m+1}(M)$ . In particular  $\sigma$  is invariant by the  $A$ -flow and the left side is finite if  $\omega$  has compact support. Thus if  $\sigma$  is a  $m+1$ - rectifiable chain with boundary  $\partial\sigma$  and  $\Omega|_{\sigma} \neq 0$  then locally  $\Pi_G(\sigma)$  is a 1-rectifiable. If we choose  $\sigma$  such that  $\Pi_A(\partial\sigma) = \{t\}$  then the left side above is a smooth function of  $t$  which vanishes outside  $\Pi_A(\text{supp } \omega)$ . We finish the proof applying the Treves propagation of zeroes theorem as we did before.

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